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ON CONSTITUTIVE RELATIONS FOR ELASTIC RODS

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Abstract—This paper explores the relationships among several existing procedures for specifying linear and nonlinear constitutive relations for hyperelastic Cosserat or directed curves from three-dimensional considerations. These procedures are designed to ensure that exact solutions of the balance laws of the directed curve are in approximate agreement with the corresponding exact solutions of classical continuum mechanics. The particular case of the linearized Bernoulli–Euler beam theory is also examined. In addition a new procedure is proposed in this paper. This procedure is used to facilitate comparisons among various existing procedures. Although it is conceptually simple, it illustrates the inherent arbitrariness in specifying constitutive relations for elastic rods.

1. INTRODUCTION

The issue of constitutive relations for a directed curve, which is a theoretical model for a rod-like body, is examined in this paper. Of particular interest are the various manners in which these relations are obtained, their relative merits and their inter-relationships. The particular theory of a directed or Cosserat curve discussed in this paper was developed by Green, Naghdi and their co-workers and dates to Green and Laws (1966). A direct approach is used there and the balance laws are stated without any obvious motivation from three-dimensional considerations. Subsequent developments, by Green, Laws and Naghdi (1967, 1968), Green and Naghdi (1970, 1979, 1985, 1990, 1993) and Green, Naghdi and Wenner (1974a, b) established the correspondances between the theory of a directed curve and three-dimensional continuum mechanics.† One of the main additional issues in these papers was the development of linear and nonlinear constitutive relations for the directed curve.

By way of additional background, we recall that in the development of the balance laws for a directed curve with two directors from the three-dimensional theory in Green *et al.* (1968, 1970, 1974a, 1985, 1993), the position vector $\mathbf{r}^*$ of a material point of a body $\mathscr{B}$ is approximated by

$$\mathbf{r}^* = \mathbf{r}^*(\theta^1, \theta^2, \xi, t) = \mathbf{r}(\xi, t) + \theta^\alpha \mathbf{d}_\alpha(\xi, t). \tag{1}$$

In equation (1) and the remainder of this paper, $\{\theta^1, \theta^2, \xi = \theta^3\}$ is a convected coordinate system which uniquely identifies the material points of $\mathscr{B}$.‡ The vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are known as directors. The approximation (1) is the basis for motivating the balance laws for the directed curve and constitutive relations for the assigned force $\mathbf{f}$, assigned director forces $\mathbf{l}^\alpha$, and inertia coefficients y^α and $y^{\alpha\beta}$.§ In addition, (1) enabled the theory of a directed curve to be successfully used by Naghdi and Rubin (1984) in their discussion of constrained rod theories, and by Naghdi and Rubin (1989) and Nordenholz and O'Reilly (1997) in applications involving contact.

† The reader is referred to the survey paper by Naghdi (1982) and the paper by Green and Naghdi (1995) for additional references and comments on this theory.

‡ In this paper, lower case Greek indices range from 1 to 2, upper case Latin indices range from 1 to 3 and the summation convention for repeated indices is employed.

§ For the reader's convenience, a summary of these and related results is presented in the Appendix.

Given the central role of (1) in applications of the theory of a directed curve, it is surprising to find that it is not used to determine the strain-energy ψ of the directed curve from the strain-energy ψ^* of the three-dimensional body. The function ψ is used to determine constitutive relations for the contact force $\mathbf{n}$, the contact director forces $\mathbf{m}^\alpha$ and the intrinsic director forces $\mathbf{k}^\alpha$ pertaining to the directed curve. Here, the main issue is to choose ψ in such a manner as to obtain a correspondence between the solutions of the balance laws for the directed curve and the balance laws for three-dimensional elasticity. However, a significant difficulty was noted on page 913 of Green *et al.* (1968) :†

> "We observe that in practise it will be impossible to compute A′ in (8.15) from the three-dimensional Helmholtz function A* ; and A′ must be regarded as an arbitrary function of its arguments."

These comments were prompted by their earlier observations in Green *et al.* (1967) that the ψ obtained by directly integrating ψ^* produces results which are in poor agreement with the exact solution for the pure flexure of a three-dimensional body. Their observations in this regard were motivated by the earlier work of Volterra (1956) and Antman and Warner (1966) where this integration was proposed.‡

It is crucial to note that the criticism of Green *et al.* (1967) applies when the displacement of the three-dimensional body that the directed curve is modeling does not satisfy a relation of the form (1). Indeed, an issue of considerable interest is the manner in which comparisons between three-dimensional continuum mechanics and the theory of a directed curve may be made when this situation arises. We will not address this issue here, but refer the reader to Section 7 of Rubin (1996) where a discussion may be found. We also take this opportunity to remark that several of Green, Naghdi and their co-workers' developments for rods were motivated by corresponding developments for shells and plates where related issues arise [cf., e.g. Sections 9 and 10 of Novozhilov (1964), Sections 16–24 of Naghdi (1972), Carroll and Naghdi (1972), Naghdi and Rubin (1995) and Reissner (1950)].

In this paper, after recording the balance laws and constitutive equations for a directed curve in Section 2, we examine three approaches for obtaining the nonlinear constitutive relations from three-dimensional considerations. Our focus is to examine the relative merits of these procedures. The first approach examined is the direct integration of ψ^*.§ In Section 3.2, a new procedure is discussed which is based on an additive decomposition. This procedure is motivated by the use of the directed curve to model situations in three-dimensional continuum mechanics where (1) does not hold. We also found that this procedure provided a convenient viewpoint for comparing the various procedures used in the literature. Specifically, we propose that

$$\psi = \psi_1 + \psi_2, \tag{2}$$

where ψ_1 is determined by integrating ψ^* and ψ_2 is specified. One criterion for the specification of the function ψ_2 is such that the solutions from the three-dimensional theory agree in an approximate sense with those for a directed curve. In Section 3.3, the approach developed by Rubin (1996) is discussed and then the inter-relationships among the three procedures are examined.

In Section 4 of this paper, the specific case of a linearly elastic rod is examined. The linearized versions of the three procedures discussed in Section 3 are first presented. In particular, the constitutive relations from these procedures are compared with results from the investigations of Green *et al.* (1967, 1974b) and Green and Naghdi (1979) on exact solutions of the balance laws for a directed curve and those for classical linear elasticity.

† In the notation of the present paper, $\psi = \text{A}'$ and $\psi^* = \text{A}^*$. Their equation (8.15) corresponds to (12) below. For additional comments on this point see page 452 of Green *et al.* (1974a) and page 127 of Green and Naghdi (1990).

‡ A discussion of the criticisms of Green *et al.* (1967) is presented in Antman (1972) [cf. also Antman and Marlow (1991)].

§ A procedure of this type for a different rod theory was discussed by Simmonds. His work is summarized in Libai and Simmonds (1988) and shows how an integration procedure can be performed using power series expansions of the functions ψ^* for various nonlinear elastic materials.

The Gibbs free-energy approach used in Green and Naghdi (1990)† and the specific case of the Bernoulli–Euler beam theory considered by Daví (1992), Dill (1992) and Love (1944) are discussed in Sections 4.4 and 4.5, respectively. The corresponding developments for the Timoshenko beam theory are easily inferred from the discussion in Section 4.

For additional background on the tensor operations and notations used in this paper, the reader is referred to Section 1.1 of Casey and Naghdi (1985) or Gurtin (1981).

2. BACKGROUND ON THE THEORY OF A DIRECTED CURVE

We recall that a directed curve $\mathscr{R}$ is a material curve $\mathscr{L}$ embedded in Euclidean three-space, together with two directors, $\mathbf{d}_1$ and $\mathbf{d}_2$, which are associated with each material point of $\mathscr{L}$. The motion of $\mathscr{R}$ is specified by three vector-valued functions

$$\mathbf{r} = \mathbf{r}(\xi, t), \quad \mathbf{d}_1 = \mathbf{d}_1(\xi, t), \quad \mathbf{d}_2 = \mathbf{d}_2(\xi, t), \tag{3}$$

where ξ is a convected coordinate and $\mathbf{r}$ is the position vector of a material point of $\mathscr{L}$ [Section 9 of Naghdi (1982)]. It is assumed that the scalar triple product $[\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3] > 0$, where $\mathbf{d}_3 = \partial\mathbf{r}/\partial\xi$. A fixed reference configuration of $\mathscr{R}$ is specified by the functions $\mathbf{R} = \mathbf{R}(\xi)$ and $\mathbf{D}_\alpha = \mathbf{D}_\alpha(\xi)$, where $[\mathbf{D}_1\mathbf{D}_2\mathbf{D}_3] > 0$ and $\mathbf{D}_3 = \partial\mathbf{R}/\partial\xi$. It is also convenient to define the reciprocal vectors $\mathbf{D}^k$ and $\mathbf{d}^k$ by $\mathbf{D}^k \cdot \mathbf{D}_i = \mathbf{d}^k \cdot \mathbf{d}_i = \delta_i^k$, where δ_i^k is the Kronecker delta.

For convenience, we employ the direct notation which was originally developed in Section 13 of Naghdi (1982). Several of his results were subsequently extended by Bechtel *et al.* (1986), O'Reilly (1995) and Rubin (1996). The invertible tensor $\mathbf{F}$ is defined by

$$\mathbf{F} = \mathbf{d}_i \otimes \mathbf{D}^i. \tag{4}$$

The additional tensors which are used to describe the deformation of $\mathscr{R}$ are

$$\mathbf{G}_\alpha = \frac{\partial \mathbf{d}_\alpha}{\partial \xi} \otimes \mathbf{D}^3, \quad {}_0\mathbf{G}_\alpha = \frac{\partial \mathbf{D}_\alpha}{\partial \xi} \otimes \mathbf{D}^3,$$
$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}), \quad \mathbf{K}_\alpha = \mathbf{F}^T\mathbf{G}_\alpha - {}_0\mathbf{G}_\alpha. \tag{5}$$

In other works, e.g. Green *et al.* (1974a), the deformation measures γ_{ij} and $\kappa_{\alpha i}$ are used. These measures are related to the tensors $\mathbf{E}$ and $\mathbf{K}_\alpha$ as follows:

$$\gamma_{ik} = \mathbf{d}_i \cdot \mathbf{d}_k - \mathbf{D}_i \cdot \mathbf{D}_k = (2\mathbf{E}\mathbf{D}_i) \cdot \mathbf{D}_k,$$
$$\kappa_{\alpha i} = \frac{\partial \mathbf{d}_\alpha}{\partial \xi} \cdot \mathbf{d}_i - \frac{\partial \mathbf{D}_\alpha}{\partial \xi} \cdot \mathbf{D}_i = (\mathbf{K}_\alpha \mathbf{D}_3) \cdot \mathbf{D}_i. \tag{6}$$

It turns out to be inconvenient for the present purposes to use $\mathbf{K}_\alpha$. Instead, the following tensors are used:

$$\boldsymbol{\lambda}_\alpha = \mathbf{F}^{-1}\mathbf{G}_\alpha = (2\mathbf{E} + \mathbf{I})^{-1}(\mathbf{K}_\alpha + {}_0\mathbf{G}_\alpha) = \left(\frac{\partial \mathbf{d}_\alpha}{\partial \xi} \cdot \mathbf{d}^i\right)\mathbf{D}_i \otimes \mathbf{D}^3. \tag{7}$$

The introduction of these tensors follows Naghdi and Rubin (1995) and Rubin (1996). They afford considerable simplification when dealing with constitutive relations. In an approximate infinitesimal theory of a directed curve whose reference configuration is such that ${}_0\mathbf{G}_\alpha = \mathbf{0}$, the approximations for $\boldsymbol{\lambda}_\alpha$ and $\mathbf{K}_\alpha$ are indistinguishable.

Associated with the contact force $\mathbf{n}$, the contact director forces $\mathbf{m}^\alpha$ and the assigned director forces $\mathbf{k}^\alpha$ are the tensors, from Section 13 of Naghdi (1982),

† An earlier version of this procedure is presented in Section 5 of Green *et al.* (1974a).

$$\sqrt{d_{33}}\mathbf{N} = \mathbf{n}\otimes\mathbf{d}_3, \quad \sqrt{d_{33}}\mathbf{K} = \mathbf{k}^\alpha\otimes\mathbf{d}_\alpha, \quad \sqrt{d_{33}}\mathbf{M}^\alpha = \mathbf{m}^\alpha\otimes\mathbf{d}_3, \tag{8}$$

where $d_{33} = \mathbf{d}_3\cdot\mathbf{d}_3$. In addition, it is convenient to define

$$\mathbf{T} = \mathbf{N}+\mathbf{K}+\mathbf{M}^\alpha(\mathbf{G}_\alpha\mathbf{F}^{-1})^{\mathrm{T}}. \tag{9}$$

Following Rubin (1996), we also define two additional tensors:

$$\sqrt{D_{33}}\mathbf{S} = \sqrt{d_{33}}\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-\mathrm{T}}, \quad \sqrt{D_{33}}\bar{\mathbf{M}}^\alpha = \sqrt{d_{33}}\mathbf{F}^{\mathrm{T}}\mathbf{M}^\alpha\mathbf{F}^{-\mathrm{T}}. \tag{10}$$

In (10), $D_{33} = \mathbf{D}_3\cdot\mathbf{D}_3$. The relation of these additional tensors to $\mathbf{n}$, $\mathbf{m}^\alpha$ and $\mathbf{k}^\alpha$ is easily inferred from (8) and (9).

The balance laws for a directed curve can be written in the form, from Section 11 of Naghdi (1982),

$$\overline{\sqrt{D_{33}}\rho_0}^{\boldsymbol{\cdot}} = 0, \quad \dot{y}^\alpha = 0, \quad \dot{y}^{\alpha\beta} = 0, \quad \frac{\partial\mathbf{n}}{\partial\xi}+\sqrt{D_{33}}\rho_0\mathbf{f} = \sqrt{D_{33}}\rho_0(\ddot{\mathbf{r}}+y^\alpha\ddot{\mathbf{d}}_\alpha),$$

$$\frac{\partial\mathbf{m}^\alpha}{\partial\xi}+\sqrt{D_{33}}\rho_0\mathbf{l}^\alpha-\mathbf{k}^\alpha = \sqrt{D_{33}}\rho_0(y^\alpha\ddot{\mathbf{r}}+y^{\alpha\beta}\ddot{\mathbf{d}}_\beta), \quad \mathbf{T} = \mathbf{T}^{\mathrm{T}}. \tag{11}$$

In (11), the superposed dot denotes the material time derivative, $\lambda = \sqrt{D_{33}}\,\rho_0$ is the mass density per unit length of the directed curve and $\rho_0 = \rho_0(\xi)$. The relations between the various quantities in (11) based on three-dimensional considerations are discussed in the Appendix.

The constitutive relations for the fields $\mathbf{n}$, $\mathbf{m}^\alpha$ and $\mathbf{k}^\alpha$ are obtained from the standard assumption†

$$\rho_0\dot{\psi} = \mathrm{tr}(\mathbf{S}\dot{\mathbf{E}}+\bar{\mathbf{M}}^\alpha\dot{\boldsymbol{\lambda}}_\alpha^{\mathrm{T}}), \tag{12}$$

where tr denotes the trace operation and

$$\psi = \psi(\mathbf{E}, \boldsymbol{\lambda}_\alpha, {}_0\mathbf{G}_\alpha, \mathbf{D}_i, \xi). \tag{13}$$

Using a standard procedure [see, e.g. Naghdi (1982), O'Reilly (1995) or Rubin (1996)] and invoking the moment of momentum balance law $(11)_6$, the constitutive relations are obtained:

$$\mathbf{S} = \rho_0\frac{\partial\psi}{\partial\mathbf{E}}, \quad \bar{\mathbf{M}}^\alpha = \rho_0\frac{\partial\psi}{\partial\boldsymbol{\lambda}_\alpha}. \tag{14}$$

The constitutive relations for $\mathbf{n}$, $\mathbf{m}^\alpha$ and $\mathbf{k}^\alpha$ can be obtained using (8)–(10).

It is convenient to recall the linearized constitutive relations for the directed curve from Green *et al.* (1974a, 1979). The reference configuration of the rod is assumed to be straight, ξ is identified with the Cartesian coordinate x_3, and $\mathbf{D}_i = \mathbf{e}_i$, where $\mathbf{e}_i$ are (orthonormal) Cartesian basis vectors. In this theory, the components of the strain tensors $2\mathbf{E}$ and $\boldsymbol{\lambda}_\alpha$ (or $\mathbf{K}_\alpha$) are approximated by

$$\gamma_{ik} = \boldsymbol{\delta}_i\cdot\mathbf{e}_k+\boldsymbol{\delta}_k\cdot\mathbf{e}_i, \quad \kappa_{\alpha i} = \frac{\partial\boldsymbol{\delta}_\alpha}{\partial x_3}\cdot\mathbf{e}_i, \tag{15}$$

respectively, where $\boldsymbol{\delta}_i = \mathbf{d}_i-\mathbf{D}_i$. The constitutive relations (14) simplify in this case to

† With the assistance of (7), these expressions follow from Eqns (2.34) and (2.40) of O'Reilly (1995).

$$n^3 = 2\lambda \frac{\partial \tilde{\psi}}{\partial \gamma_{33}}, \quad n^\alpha = \lambda \frac{\partial \tilde{\psi}}{\partial \gamma_{\alpha 3}}, \quad \frac{1}{2}(k^{\alpha\beta} + k^{\beta\alpha}) = 2\lambda \frac{\partial \tilde{\psi}}{\partial \gamma_{\alpha\beta}}, \quad m^{\alpha i} = \lambda \frac{\partial \tilde{\psi}}{\partial \kappa_{\alpha i}}, \tag{16}$$

where $n^i = \mathbf{n} \cdot \mathbf{e}_i$, $m^{\alpha i} = \mathbf{m}^\alpha \cdot \mathbf{e}_i$ and $k^{\alpha i} = \mathbf{k}^\alpha \cdot \mathbf{e}_i$. These relations are supplemented by the linearized moment of momentum balance law:

$$k^{\alpha 3} = n^\alpha, \quad k^{12} = k^{21}. \tag{17}$$

In (16), the strain energy $\tilde{\psi} = \tilde{\psi}(\gamma_{i3}, (1/2)(\gamma_{\alpha\beta} + \gamma_{\beta\alpha}), \kappa_{\alpha i}, x_3)$ where we have used a tilde to distinguish the linearized strain-energy $\tilde{\psi}$ and the nonlinear strain-energy ψ. Alternatively, if the Gibbs free energy ϕ is used, then the relations corresponding to (16) are

$$\gamma_{33} = 2\lambda \frac{\partial \phi}{\partial n^3}, \quad \gamma_{\alpha 3} = \lambda \frac{\partial \phi}{\partial n^\alpha}, \quad \frac{1}{2}(\gamma_{\alpha\beta} + \gamma_{\beta\alpha}) = 2\lambda \frac{\partial \phi}{\partial k^{\alpha\beta}}, \quad \kappa_{\alpha i} = \lambda \frac{\partial \phi}{\partial m^{\alpha i}}, \tag{18}$$

where $\phi = \phi(n^i, (1/2)(k^{\alpha\beta} + k^{\beta\alpha}), m^{\alpha i}, x_3)$. The Gibbs function and strain-energy function are related by a Legendre transformation defined by (16) and

$$\lambda \phi = \frac{1}{2} n^3 \gamma_{33} + n^\alpha \gamma_{\alpha 3} + \frac{1}{4}(\gamma_{\alpha\beta} + \gamma_{\alpha\beta})(k^{\alpha\beta} + k^{\beta\alpha}) + m^{\alpha i} \kappa_{\alpha i} - \lambda \tilde{\psi}. \tag{19}$$

The definition of ϕ provided by (19) differs from the one used in Green *et al.* (1974a) by a multiplicative factor of -1.

3. SPECIFICATIONS FOR THE STRAIN-ENERGY OF A DIRECTED CURVE: THE NONLINEAR CASE

In this section, we examine specifications for ψ based on three-dimensional considerations. The first of these specifications is the direct integration mentioned earlier. Next, a new specification is discussed which is based on the additive decomposition of ψ. This specification is also related to the direct integration procedure. The third specification was recently proposed by Rubin (1996). It is intimately related to earlier work on shells and plates by Naghdi and Rubin (1995), and we will show how it is related to the new specification proposed here. The relative merits of the various procedures are also discussed, although this issue is hindered by the lack of exact solutions from three-dimensional continuum mechanics that are presently available.

3.1. *The direct integration approach*

To examine the direct integration approach, it is appropriate to recall that the balance laws for the directed curve were motivated by three-dimensional considerations [cf., e.g. Green and Naghdi (1970, 1993)]. Several of these results are recalled in the Appendix. To specify ψ by direct integration, it is convenient to retrace the development of the expression for the mechanical power by Green and Naghdi (1970, 1985, 1993) and Green *et al.* (1974).

As noted in the Appendix, to model the deformation of a three-dimensional body $\mathscr{B}$ using the directed curve, one chooses the convected coordinates θ^i such that the following representation holds in the fixed reference configuration $\boldsymbol{\kappa}_0$ of $\mathscr{B}$:

$$\mathbf{R}^* = \mathbf{R}(\xi) + \theta^\alpha \mathbf{D}_\alpha(\xi). \tag{20}$$

It follows from (A1) that

$$\mathbf{G}^*_k \otimes \mathbf{D}^k = \mathbf{I} + \theta^\alpha{}_0\mathbf{G}_\alpha, \quad \mathbf{G}^{*k} \otimes \mathbf{D}_k = (\mathbf{I} + \theta^\alpha{}_0\mathbf{G}_\alpha)^{-T}. \tag{21}$$

After substituting the approximation (1) for $\mathbf{r}^*$ and the representation (20) in (A1), it may be seen that the three-dimensional deformation gradient $\mathbf{F}^* = \mathbf{g}_i^* \otimes \mathbf{G}^{i*}$ is then approximated by $\tilde{\mathbf{F}}^*$:

$$\tilde{\mathbf{F}}^* = \mathbf{F}\mathbf{A}, \tag{22}$$

where

$$\mathbf{A} = (\mathbf{I} + \theta^\sigma \boldsymbol{\lambda}_\sigma)(\mathbf{I} + \theta^\alpha {}_0\mathbf{G}_\alpha)^{-1}. \tag{23}$$

The corresponding approximation for the three-dimensional Lagrangian strain tensor $\mathbf{E}^* = (1/2)((\mathbf{F}^*)^{\mathrm{T}}\mathbf{F}^* - \mathbf{I})$, which we denote by $\tilde{\mathbf{E}}^*$, is obtained by substituting (22) for $\mathbf{F}^*$ into the definition of $\mathbf{E}^*$:

$$\tilde{\mathbf{E}}^* = \mathbf{A}^{\mathrm{T}}\mathbf{E}\mathbf{A} + \frac{1}{2}(\mathbf{A}^{\mathrm{T}}\mathbf{A} - \mathbf{I}). \tag{24}$$

The material time derivative of $\tilde{\mathbf{E}}^*$ is obtained after a long, but straightforward, calculation and, in the interests of brevity, is not recorded here.

We recall that the constitutive relations for a hyperelastic body are obtained from a work-energy theorem:

$$\int_{\xi_1}^{\xi_2} \int_{\mathscr{A}} \rho_0^* \dot{\psi}^*(\mathbf{E}^*, \mathbf{G}_i^* \cdot \mathbf{G}_k^*, \theta^i)\, \mathrm{d}V = \int_{\xi_1}^{\xi_2} \int_{\mathscr{A}} \mathrm{tr}(\mathbf{S}^*, \dot{\mathbf{E}}^*)\, \mathrm{d}V, \tag{25}$$

where ψ^* is the strain energy density function, ρ_0^* is the mass density and $\mathbf{S}^*$ is the second (or symmetric) Piola–Kirchhoff stress tensor. The material surface $\mathscr{A}$, and the coordinates ξ_1 and ξ_2 are defined in the Appendix. Using (24), the mechanical power of $\mathscr{B}$ is approximated by

$$\int_{\xi_1}^{\xi_2} \int_{\mathscr{A}} \mathrm{tr}(\mathbf{S}^* \dot{\tilde{\mathbf{E}}}^*)\, \mathrm{d}V = \int_{\xi_1}^{\xi_2} \mathrm{tr}(\mathbf{S}\dot{\mathbf{E}} + \bar{\mathbf{M}}^\alpha \dot{\boldsymbol{\lambda}}_\alpha^{\mathrm{T}}) \sqrt{D_{33}}\, \mathrm{d}\xi, \tag{26}$$

where

$$\sqrt{D_{33}}\mathbf{S} = \int_{\mathscr{A}} \mathbf{A}\mathbf{S}^*\mathbf{A}^{\mathrm{T}} \sqrt{G^*}\, \mathrm{d}\theta^1\, \mathrm{d}\theta^2,$$

$$\sqrt{D_{33}}\bar{\mathbf{M}}^\alpha = \int_{\mathscr{A}} \mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{A}\mathbf{S}^*\mathbf{G}^{*3} \theta^\alpha \sqrt{G^*}\, \mathrm{d}\theta^1\, \mathrm{d}\theta^2 \otimes \mathbf{D}_3. \tag{27}$$

In (27), $\sqrt{\mathbf{G}^*} = [\mathbf{G}_1^*\mathbf{G}_2^*\mathbf{G}_3^*]$. The results (27) are in expected agreement both with $(\mathrm{A9})_{4,5,6}$ and (A10). Provided one considers the approximation (22) for $\mathbf{F}^*$, they are also in expected agreement with results in Green and Naghdi (1970, 1985, 1993) and Green *et al.* (1974a) where the Cauchy stress tensor $\mathbf{T}^*$ is used. Furthermore, the right-hand sides of (26) and (12) are in expected agreement.

Using (A1), (20)–(22) and the local form of (26), in the direct integration procedure ψ of the directed curve is specified by $\psi = \psi_1$, where

$$\lambda\psi_1 = \int_{\mathscr{A}} \rho_0^* \psi(\mathbf{E}^* = \tilde{\mathbf{E}}^*, {}_0\mathbf{G}_\alpha, \mathbf{D}_i, \theta^i) \sqrt{G^*}\, \mathrm{d}\theta^1\, \mathrm{d}\theta^2. \tag{28}$$

Clearly ψ_1 has the functional form (13), and it is tacitly assumed that the integral in (28) is well defined. It also should be noted that $G_{ik}^* = \mathbf{G}_i^* \cdot \mathbf{G}_k^*$ was expressed as a function of $\mathbf{D}_i$, ${}_0\mathbf{G}_\alpha$ and θ^i in writing (28).

If the deformation of $\mathscr{B}$ is such that $\mathbf{F}^* = \mathbf{FA}$, then the specification (28) and the constitutive relations it provides [using (14)] for $\mathbf{n}$, $\mathbf{m}^\alpha$ and $\mathbf{k}^\alpha$ are completely consistent with the balance laws for the directed curve when the identifications recorded in the Appendix are used. However, we know of no exact solutions of three-dimensional continuum mechanics of the form $\mathbf{F}^* = \mathbf{FA}$. Furthermore, for many problems where it is of interest to use the theory of a directed curve $\mathbf{F}^* \neq \mathbf{FA}$: an issue which, as Green *et al.* (1967) have pointed out, clearly presents itself in the context of the linear theory.

3.2. *An additive decomposition of the strain-energy*

In contrast to previous works, we propose here to specify ψ by an additive decomposition:

$$\lambda\psi = \lambda\psi_1 + \lambda\psi_2, \tag{29}$$

where ψ_1 is defined by (28) and ψ_α have the functional form (13).† The function ψ_2 is subject to a restriction: when $\mathbf{F}^* = \mathbf{FA}$, then $\psi_2 = 0$. Furthermore, the specification (29) is consistent with the work-energy theorem of the three-dimensional theory only when $\lambda\dot{\psi}_2 = 0$. Once ψ_2 has been specified, the constitutive relations for $\mathbf{n}$, $\mathbf{m}^\alpha$ and $\mathbf{k}^\alpha$ can be obtained using (29) and (14).

The presence of the function ψ_2 is motivated by applications of the theory of a directed curve to the modeling of three-dimensional continuua where $\mathbf{F}^* \neq \mathbf{FA}$. This situation arises when the directed curve is used to model the flexure of a linearly elastic body.‡ In these cases, an approximation procedure is used to obtain the constitutive relations for $\mathbf{n}$, $\mathbf{m}^\alpha$ and $\mathbf{k}^\alpha$. As will be illustrated later, several procedures of this type are formally equivalent to the specification (29).

For future purposes, we note that we may approximate $\mathbf{E}^*$ by $\tilde{\mathbf{E}}^*$ in the constitutive relation $\mathbf{S}^* = \rho_0^*(\partial\psi^*/\partial\mathbf{E}^*)$ to determine the corresponding approximation $\tilde{\mathbf{S}}^*$ to $\mathbf{S}^*$:

$$\tilde{\mathbf{S}}^* = \tilde{\mathbf{S}}^*(\mathbf{E}, \boldsymbol{\lambda}_\alpha, {}_0\mathbf{G}_\alpha, \mathbf{D}_i, \theta^i). \tag{30}$$

In addition, assuming that the constitutive relations for $\mathbf{S}$ and $\bar{\mathbf{M}}^\alpha$ can be inverted, at least locally, to determine $\mathbf{E}$ and $\boldsymbol{\lambda}_\alpha$ as functions of $\mathbf{S}$ and $\bar{\mathbf{M}}^\alpha$:

$$\mathbf{E} = \hat{\mathbf{E}}(\mathbf{S}, \bar{\mathbf{M}}^\alpha, {}_0\mathbf{G}_\alpha, \mathbf{D}_i, \xi), \quad \boldsymbol{\lambda}_\alpha = \hat{\boldsymbol{\lambda}}_\alpha(\mathbf{S}, \bar{\mathbf{M}}^\alpha, {}_0\mathbf{G}_\alpha, \mathbf{D}_i, \xi). \tag{31}$$

Using (30) and (31), one can obtain the functions

$$\tilde{\mathbf{S}}^* = \bar{\tilde{\mathbf{S}}}^*(\mathbf{S}, \bar{\mathbf{M}}^\alpha, {}_0\mathbf{G}_\alpha, \mathbf{D}_i, \theta^i). \tag{32}$$

By construction, these functions will satisfy (27) when $\psi_2 = 0$. Moreover, they are clearly influenced by ψ_1 and ψ_2. Indeed in Section 4.2, the resulting functions in a linear theory will not satisfy (27) for a specific $\psi_2 \neq 0$.

The merit of the specification (29) is that it ensures that the balance laws for the directed curve are satisfied if the motion of a three-dimensional body is such that $\mathbf{F}^* = \mathbf{FA}$: an advantage this specification shares with the one discussed in Section 3.1. However, a feasible procedure for calculating ψ_2 for non-linearly elastic bodies remains unknown. This is partially attributable to our lack of knowledge of exact solutions of the form $\mathbf{F}^* = \mathbf{FA}$.

3.3. *The specification of Rubin*

Rubin (1996) considers restrictions on ψ such that exact solutions of the balance laws for the directed curve are consistent with exact solutions for homogeneous deformations

† It is manifestly possible to specify ψ by a multiplicative decomposition, although we do not explore such specifications here.

‡ As noted on page 297 of Green *et al.* (1967), it is not possible to model the three-dimensional displacement fields for the pure flexure of a rod using (1) and (20). Related comments are contained in Section 11 of Antman (1972) and Chapter 6 of Novozhilov (1953).

of homogeneous bodies. A homogeneous deformation of $\mathscr{B}$ is such that $\mathbf{F}^* = \mathbf{F}^*(t)$. It follows from (20)–(23) that the corresponding deformation for a directed curve is $\mathbf{F} = \mathbf{F}^*(t)$ and $\mathbf{A} = \mathbf{I}$. Consequently, Rubin's restrictions for elastic rods are

$$\lambda \frac{\partial \psi}{\partial \mathbf{E}}\bigg|_{\lambda_\alpha = {}_0G_\alpha} = \rho_0^* \frac{\partial \psi^*}{\partial \mathbf{E}^*}\int_{\mathscr{A}} \sqrt{G^*}\, d\theta^1\, d\theta^2,$$

$$\lambda \frac{\partial \psi}{\partial \boldsymbol{\lambda}_\beta}\bigg|_{\lambda_\alpha = {}_0G_\alpha} = \rho_0^*(2\mathbf{E}+\mathbf{I}) \frac{\partial \psi^*}{\partial \mathbf{E}^*}\int_{\mathscr{A}} \mathbf{G}^{*3}\theta^\beta \sqrt{G^*}\, d\theta^1\, d\theta^2 \otimes \mathbf{D}_3. \tag{33}$$

These restrictions motivated Rubin to consider the following decomposition of ψ.†

$$\psi = \psi^*(\mathbf{E}^* = \bar{\mathbf{E}}, {}_0\mathbf{G}_\alpha, \mathbf{D}_i) + \Psi(\mathbf{E}, \boldsymbol{\lambda}^\alpha, {}_0\mathbf{G}_\alpha, \mathbf{D}_i, \xi), \tag{34}$$

where

$$2\bar{\mathbf{E}} = (\mathbf{I} + A^\alpha(\boldsymbol{\lambda}_\alpha - {}_0\mathbf{G}_\alpha))^{\mathrm{T}}(2\mathbf{E}+\mathbf{I})(\mathbf{I} + A^\beta(\boldsymbol{\lambda}_\beta - {}_0\mathbf{G}_\beta)) - \mathbf{I}, \tag{35}$$

and A^α are defined by (A13). The function Ψ is subject to the restrictions

$$\lambda \frac{\partial \Psi}{\partial \mathbf{E}}\bigg|_{\lambda_\alpha = {}_0G_\alpha} = \mathbf{0}, \quad \lambda \frac{\partial \Psi}{\partial \boldsymbol{\lambda}_\beta}\bigg|_{\lambda_\alpha = {}_0G_\alpha} = \mathbf{0}. \tag{36}$$

These restrictions ensure that ψ satisfies (33).

Rubin's procedure has the significant advantage that it is based on solutions for three-dimensional continuum mechanics which have been extensively studied [cf., e.g. Truesdell and Noll (1992)]. In addition, as proved by Ericksen (1955), static, homogeneous deformations are the only controllable, static deformations possible in every homogeneous, isotropic, hyperelastic body.‡ In this light, Rubin's work may be viewed as a bridge between the experimental determination of constitutive relations in three-dimensional continuum mechanics and the specification of the corresponding quantities for a directed curve.

It is clearly of interest to compare Rubin's specification with the two others discussed earlier. First, if $\psi = \psi_1$ [cf. (28)], then ψ trivially satisfies the restrictions (33). Secondly, because $\psi_2 = 0$ for homogeneous deformations, the specification (29) also satisfies the restrictions (33). It is also of interest to note that, as in the specification (29), only part of ψ is determined from ψ^* by (34). Finally, the specification (34) shares a disadvantage of (29) : there are numerous possible choices of Ψ which satisfy (36).

4. SPECIFICATIONS FOR THE STRAIN-ENERGY OF A DIRECT CURVE: THE LINEARLY ELASTIC ROD

In this section, we specialize the discussion of Section 3 to the case where $\mathscr{B}$ is composed of an isotropic, linearly elastic material. In addition, several other procedures for determining the linearized strain-energy $\tilde{\psi}$ are discussed. One of these is the Gibbs free energy approach developed by Green and Naghdi (1990). The others pertain to Bernoulli–Euler beams and were discussed by Daví (1992), Dill (1992) and Love (1944), among others. For convenience, the tilde which distinguished the linearized strain-energy in Section 2 is omitted henceforth.

† These results are from Section 4 of Rubin (1996). To facilitate comparisons with the results of the present paper, we have made some minor notational changes to his results.

‡ A controllable deformation is a deformation which is sustained by the application of surface tractions alone. Beatty (1987) has an interesting discussion of their importance. Recently, Faulkner and Steigmann (1993) have discussed deformations of this type for a particular constrained rod theory.

4.1. *The direct integration approach*

We first consider the linearized version of the procedure discussed in Section 3.1. In so doing, several preliminary results which will be used in the subsequent subsections are also established. Several of these results and the integration procedure itself are contained in the papers of Green *et al.* (1967, 1974a) and are reproduced here in the interests of future comprehension.

For the linear theory of interest, the convected coordinates $\{\theta^i\}$ are chosen to be a Cartesian coordinate system $\{x_i\}$ and $\mathbf{D}_i = \mathbf{e}_i$. The three-dimensional strain-energy for this case is [cf., e.g. Section 26 of Sokolnikoff (1956)],

$$\rho_0^*\psi^*(e_{ik}^*) = \frac{E\nu}{2(1+\nu)(1-2\nu)}(e_{11}^*+e_{22}^*+e_{33}^*)^2, + \frac{E}{2(1+\nu)}(e_{11}^{*2}+e_{22}^{*2}+e_{33}^{*2}+2e_{12}^{*2}+2e_{23}^{*2}+2e_{13}^{*2}), \tag{37}$$

where E is Young's modulus and ν is Poisson's ratio. In conjunction with (37), the Gibbs free energy ϕ^* can be defined in the standard manner using a Legendre transformation:†

$$\rho_0^*\phi^*(\tau_{ik}^*) = \tau_{in}^* e_{in}^* - \rho_0^*\psi^*(e_{ik}^*), \tau_{in}^* = \rho_0^*\frac{\partial\psi^*}{\partial e_{in}^*}. \tag{38}$$

It follows from (38) that

$$\rho_0^*\phi^*(\tau_{ik}^*) = -\frac{\nu}{2E}(\tau_{11}^*+\tau_{12}^*+\tau_{33}^*)^2 + \frac{(1+\nu)}{2E}(\tau_{11}^{*2}+\tau_{22}^{*2}+\tau_{33}^{*2}+2\tau_{12}^{*2}+2\tau_{23}^{*2}+2\tau_{13}^{*2}). \tag{39}$$

In (38) and (39), τ_{in}^* are the components of a (symmetric) stress tensor: $\mathbf{S}^* = \tau_{in}^*\mathbf{e}_i \otimes \mathbf{e}_n$. The linear strains measures e_{ik}^* and displacements u_k^* are

$$e_{ik}^* = \frac{1}{2}\left(\frac{\partial u_i^*}{\partial x_k} + \frac{\partial u_k^*}{\partial x_i}\right), \quad u_k^* = (\mathbf{r}^* - \mathbf{R}^*)\cdot\mathbf{e}_k. \tag{40}$$

Finally, it is assumed that the reference configuration of $\mathscr{B}$ has two planes of symmetry: $x_\alpha = 0$.

We are now in a position to calculate ψ_1 for the directed curve by paralleling the developments in Section 3.1. The only differences are that γ_{ik}^* are replaced by e_{ik}^* and (15) replaces (5) and (7):

$$\tilde{e}_{11}^* = \frac{1}{2}\gamma_{11}, \quad \tilde{e}_{22}^* = \frac{1}{2}\gamma_{22}, \quad \tilde{e}_{33}^* = \frac{1}{2}\gamma_{33} + x_\alpha\kappa_{\alpha 3}.$$
$$\tilde{e}_{12}^* = \frac{1}{2}\gamma_{12}, \quad \tilde{e}_{13}^* = \frac{1}{2}\gamma_{13} + \frac{1}{2}x_\alpha\kappa_{\alpha 1}, \quad \tilde{e}_{23}^* = \frac{1}{2}\gamma_{23} + \frac{1}{2}x_\alpha\kappa_{\alpha 2}. \tag{41}$$

After a long calculation using (28), (37) and (41), one obtains

† The difference between (39) and eqn (5.3) of Green *et al.* (1974a) is due to a trivial difference in the Legendre transformations used.

$$2\lambda\psi_1 = \bar{k}_1\gamma_{11}^2+\bar{k}_2\gamma_{22}^2+\bar{k}_3\gamma_{33}^2+\bar{k}_7\gamma_{11}\gamma_{22}+\bar{k}_8\gamma_{11}\gamma_{33}+\bar{k}_9\gamma_{22}\gamma_{33}$$
$$+\frac{\bar{k}_4}{4}(\gamma_{12}+\gamma_{21})^2+\bar{k}_5\gamma_{23}^2+\bar{k}_6\gamma_{13}^2+\bar{k}_{10}\kappa_{11}^2+\bar{k}_{11}\kappa_{22}^2+\bar{k}_{12}\kappa_{12}^2+\bar{k}_{13}\kappa_{21}^2$$
$$+\bar{k}_{14}\kappa_{12}\kappa_{21}+\bar{k}_{15}\kappa_{23}^2+\bar{k}_{16}\kappa_{13}^2+\bar{k}_{17}\kappa_{11}\kappa_{22}, \quad (42)$$

where

$$\bar{k}_1 = \bar{k}_2 = \bar{k}_3 = \frac{\mu A(1-\nu)}{2(1-2\nu)}, \quad \bar{k}_4 = \bar{k}_5 = \bar{k}_6 = \mu A, \quad \bar{k}_7 = \bar{k}_8 = \bar{k}_9 = \frac{\mu A\nu}{(1-2\nu)},$$
$$\bar{k}_{10} = \bar{k}_{12} = \mu I_2, \quad \bar{k}_{11} = \bar{k}_{13} = \mu I_1,$$
$$\bar{k}_{14} = 0, \bar{k}_{15} = \frac{EI_1(1-\nu)}{(1+\nu)(1-2\nu)}, \quad \bar{k}_{16} = \frac{EI_2(1-\nu)}{(1+\nu)(1-2\nu)}, \quad \bar{k}_{17} = 0. \quad (43)$$

In addition, $\mu = E/(2(1+\nu))$ is one of Lamé's constants and

$$A = \int_{\mathscr{A}} \mathrm{d}x_1\,\mathrm{d}x_2, \quad I_1 = \int_{\mathscr{A}} x_2^2\,\mathrm{d}x_1\,\mathrm{d}x_2, \quad I_2 = \int_{\mathscr{A}} x_1^2\,\mathrm{d}x_1\,\mathrm{d}x_2. \quad (44)$$

The quantities I_α are the moments of area with respect to the x_α axes.

It will be of future interest to examine the approximations to the components of $\mathbf{S}^*$ [cf. (30)–(32)] which result from using the specification $\lambda\psi = \lambda\psi_1$. First, the constitutive relations for $\mathbf{n}$, $\mathbf{m}^\alpha$ and $\mathbf{k}^\alpha$ are determined using (16), (17) and (42). These relations are then inverted to determine γ_{ij} and $\kappa_{\alpha i}$ as functions of n^i, $m^{\alpha i}$ and $k^{\alpha i}$. An additional calculation which uses (37), (38) and (42) determines τ_{ik}^* as functions of γ_{ij} and $\kappa_{\alpha i}$. To simplify the resulting expressions for τ_{11}^*, τ_{22}^*, and τ_{33}^*, the following identities are used:

$$\bar{k}_1 = \bar{k}_2 = \bar{k}_3 = \frac{\mu A(1-\nu)}{2(1-2\nu)}, \quad \bar{k}_7 = \bar{k}_8 = \bar{k}_9 = \frac{2\nu}{(1-\nu)}\bar{k}_1, \quad \bar{k}_{17} = 0. \quad (45)$$

However, the specifications (43) are not used for the remaining coefficients. Combining the results of the aforementioned calculations yields

$$\tau_{12}^* = \frac{\mu}{2\bar{k}_4}(k^{12}+k^{21}),$$
$$\tau_{13}^* = \frac{\mu}{\bar{k}_6}n^1+\frac{\mu x_1}{\bar{k}_{10}}m^{11}+\frac{\mu x_2}{\bar{k}_{12}\bar{k}_{13}-0.25\bar{k}_{14}^2}(\bar{k}_{13}m^{12}-0.5\bar{k}_{14}m^{21}),$$
$$\tau_{23}^* = \frac{\mu}{\bar{k}_5}n^2+\frac{\mu x_1}{\bar{k}_{12}\bar{k}_{13}-0.25\bar{k}_{14}^2}(\bar{k}_{12}m^{21}-0.5\bar{k}_{14}m^{12}+\frac{\mu x_2}{\bar{k}_{11}}m^{22},$$
$$\tau_{\alpha\alpha}^* = \frac{k^{\alpha\alpha}}{A}+\left\{\frac{2\mu\nu}{(1-2\nu)}\left(\frac{x_1 m^{13}}{\bar{k}_{16}}+\frac{x_2 m^{23}}{\bar{k}_{15}}\right)\right\} \quad \text{(no sum on } \alpha\text{)},$$
$$\tau_{33}^* = \frac{n^3}{A}+\frac{2\mu(1-\nu)}{(1-2\nu)}\left(\frac{x_1 m^{13}}{\bar{k}_{16}}+\frac{x_2 m^{23}}{\bar{k}_{15}}\right). \quad (46)$$

Another calculation using the values of $\bar{k}_1, \ldots, \bar{k}_{16}$ provided in (43) shows that functions τ_{ik}^* are in accord with the linearized versions of (27) and $(\mathrm{A9})_{4,5,6}$, from Section 5 of Green *et al.* (1974a),

$$n^i = \int_{\mathscr{A}} \tau^*_{3i}\,\mathrm{d}x_1\,\mathrm{d}x_2, \quad m^{\alpha i} = \int_{\mathscr{A}} \tau^*_{3i}x_\alpha\,\mathrm{d}x_1\,\mathrm{d}x_2, k^{\alpha i} = \int_{\mathscr{A}} \tau^*_{\alpha i}\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{47}$$

The terms of $(46)_4$ in the curly brackets {} are such that their contributions to the integrals in (47) are zero.

The specifications of $\bar{k}_1, \ldots, \bar{k}_{17}$ for homogeneous, isotropic, linearly elastic bodies of constant cross-sectional area were determined in a series of works by Green *et al.* (1967, 1974a) and Green and Naghdi (1979) using various comparisons with exact solutions from three-dimensional elasticity. We summarize them here:

$$k_1 = k_2 = k_3 = \frac{\mu A(1-\nu)}{2(1-2\nu)}, \quad k_4 = \mu A, \quad k_5 = k_6 = EAf(\nu), \quad k_7 = k_8 = k_9 = \frac{\mu A\nu}{(1-2\nu)},$$

$$k_{10} = \mu I_2, \quad k_{11} = \mu I_1, \quad k_{12} = k_{13} = \frac{1}{4}(\mu I_1 + \mu I_2 + \mathscr{D}),$$

$$k_{14} = \frac{1}{2}(\mu I_1 + \mu I_2 - \mathscr{D}), \quad k_{15} = EI_1, \quad k_{16} = EI_2, \quad k_{17} = 0, \tag{48}$$

where, $\mathscr{D}$ is the torsional rigidity, and we have dropped the overbars to distinguish these results from those obtained using $\lambda\psi_1$. The function $f(\nu)$ has numerous values in the literature. In particular, Rubin (1996) proposed that $f(\nu) = \mu/E$, while Naghdi and Rubin (1989) used $f(\nu) = 5\mu/6E$. These choices are equivalent to assuming shear coefficients of 1 and 5/6, respectively.

If $\lambda\psi = \lambda\psi_1$, then a problem noted by Green *et al.* (1967) becomes evident. Specifically, the flexural rigidities $\bar{k}_{15}$ and $\bar{k}_{16}$ (as predicted by $\lambda\psi_1$) differ considerably from k_{15} and k_{16}. For the torsion of a rod which does not have a constant cross-sectional radius R, a similar discrepancy arises between the coefficients $\bar{k}_{12}$, $\bar{k}_{13}$, $\bar{k}_{14}$ and k_{12}, k_{13}, k_{14}. Consequently, exact solutions from three-dimensional linear elasticity and those for the linearly elastic directed curve will be in poor agreement. However, it is important to note that several of the coefficients in (43) are identical to those in (48). In particular, $\bar{k}_{10}$, $\bar{k}_{11}$ and $\bar{k}_{17}$ are in agreement with k_{10}, k_{11} and k_{17}. The latter coefficients were obtained by Green and Naghdi (1979). An examination of the solution they used to establish these results [cf eqn (10.38) of their paper] shows that the three-dimensional displacement fields are in agreement with (1) and (20). We shall provide additional comment on the other coefficients later.

4.2. *An additive decomposition of the strain-energy*

The value of $\lambda\psi_1$ used for the specification (29) is given by (42). To resolve the discrepancies noted above, it suffices to specify

$$2\lambda\psi_2 = (k_5 - \bar{k}_5)\gamma_{23}^2 + (k_6 - \bar{k}_6)\gamma_{13}^2 + (k_{12} - \bar{k}_{12})\kappa_{12}^2 + (k_{13} - \bar{k}_{13})\kappa_{21}^2 + k_{14}\kappa_{12}\kappa_{21} + (k_{15} - \bar{k}_{15})\kappa_{23}^2 + (k_{16} - \bar{k}_{16})\kappa_{13}^2. \tag{49}$$

Although this provides the correct coefficients $k_1, \ldots, k_{17}$, the deficiency inherent in the selection $\lambda\psi = \lambda\psi_1$ must be known *a priori*.

With the specification $\lambda\psi = \lambda\psi_1 + \lambda\psi_2$, where $\lambda\psi_1$ is given by (42) and $\lambda\psi_2$ is given by (49), the expressions for the approximations to the components of $\mathbf{S}^*$ become:

$$\tau^*_{12} = \frac{1}{2A}(k^{12} + k^{21}),$$

$$\tau^*_{13} = \frac{\mu}{EAf(\nu)}n^1 + \frac{x_1}{I_2}m^{11} + \frac{\mu x_2}{k_{12}k_{13} - 0.25k_{14}^2}(k_{13}m^{12} - 0.5k_{14}m^{21}),$$

$$\tau_{23}^* = \frac{\mu}{EAf(v)}n^2 + \frac{\mu x_1}{k_{12}k_{13} - 0.25k_{14}^2}(k_{12}m^{21} - 0.5k_{14}m^{12}) + \frac{\mu x_2}{I_1}m^{22},$$

$$\tau_{\alpha\alpha}^* = \frac{k^{\alpha\alpha}}{A} + \left\{\frac{2\mu v}{E(1-2v)}\left(\frac{x_1 m^{13}}{I_2} + \frac{x_2 m^{23}}{I_1}\right)\right\} \quad \text{(no sum on } \alpha),$$

$$\tau_{33}^* = \frac{n^3}{A} + \frac{2\mu(1-v)}{E(1-2v)}\left(\frac{x_1 m^{13}}{I_2} + \frac{x_2 m^{23}}{I_1}\right). \tag{50}$$

These results can be obtained from (46) by first dropping the overbars on $\bar{k}_1, \ldots, \bar{k}_{16}$ and then using (48). However, (50) does not satisfy (47). Specifically, the equations for the resultants n^1, n^2, m^{13} and m^{23} associated with flexure, and m^{21} and m^{12} associated with torsion are generally not satisfied.

4.3. *The specification of Rubin*

In Section 6 of Rubin (1996), the case of a straight rod similar to that discussed in this paper is considered. In consonance with the restrictions (33), $\lambda\psi$ is specified as an additive decomposition of the form (34). Rubin's results for the coefficients $\bar{k}_1$, $\bar{k}_2$, $\bar{k}_3$, $\bar{k}_4$, $\bar{k}_5$, $\bar{k}_6$, $\bar{k}_7$, $\bar{k}_8$, $\bar{k}_9$, are, as expected, identical.† This agreement also lends further support to Rubin's assertion that the shear coefficient should be 1. However, in order to specify all of the coefficients, Rubin was forced to use the results of Green *et al.* (1967) and Green and Naghdi (1979) for k_{10}, k_{11}, k_{15}, k_{16} and k_{17}. In other words, the function Ψ in (34) is not completely specified by his restrictions and further recourse to solutions from three-dimensional elasticity is necessary. In this context, he also notes that there is an inherent arbitrariness in comparing solutions of the two theories in these cases. Finally, his results for the linear case are equivalent to the specification (49) for $\lambda\psi_2$ provided $f(v) = \mu/E$.

4.4. *The Gibbs free-energy approach of Green and Naghdi*

We now comment on the Gibbs free energy approach used by Green and Naghdi (1990). An earlier version of this procedure is presented in Section 5 of Green *et al.* (1974a). In this approach, which is used in a linear theory,‡ the approximation (1) and, in addition, approximations of the form $\tau^{ij} = \tau^{ij}(n^l, m^{\alpha l}, k^{\alpha l})$ are used. From eqn (2.15) of Green and Naghdi (1990), these approximations are

$$\tau^{12} = \frac{1}{2A}(k^{12} + k^{21}), \quad \tau^{\alpha 3} = \frac{n^\alpha}{A} + x_1\frac{m^{1\alpha}}{I_2} + x_2\frac{m^{2\alpha}}{I_1},$$

$$\tau^{\alpha\alpha} = \frac{k^{\alpha\alpha}}{A} \quad \text{(no sum on } \alpha), \quad \tau^{33} = \frac{n^3}{A} + x_1\frac{m^{13}}{I_2} + x_2\frac{m^{23}}{I_1}. \tag{51}$$

It should be noted that (51), by design, satisfy (47) and are in fact identical to (46) if the terms in the curly brackets {} are ignored and (43) is used.

The approximations (51) may be substituted into (39) and the result integrated over the material surface $\mathscr{A}(\xi)$. The resulting function is known as the Gibbs free energy ϕ_{GN} of the directed curve:

$$\begin{aligned}2\lambda\phi_{GN} = {} & \frac{1}{EA}((n^3)^2 + (k^{11})^2 + (k^{22})^2 - \frac{2v}{EA}(n^3k^{11} + n^3k^{22} + k^{11}k^{22}) \\ & + \frac{1}{EI_1}(m^{23})^2 + \frac{1}{\mu I_1}((m^{21})^2 + (m^{22})^2) + \frac{1}{\mu I_2}((m^{11})^2 + (m^{12})^2) \\ & + \frac{1}{EI_2}(m^{13})^2 + \frac{1}{\mu A}((n^1)^2 + (n^2)^2 + \frac{1}{4}(k^{12} + k^{21})^2).\end{aligned} \tag{52}$$

† As may be seen from (41), for a homogeneous deformation $\kappa_{\alpha i} = 0$. The nine coefficients associated with the non-trivial strains γ_{ij} can be ascertained from (42) by inspection.

‡ We were unable to generalize Green and Naghdi's Gibbs free energy approach to the nonlinear theory.

With the assistance of (18) and (19), the strain-energy $\lambda\psi_{GN}$ corresponding to (52) can be calculated. Omitting details, the resulting expression for $\lambda\psi_{GN}$ can be obtained from (42) by replacing $\bar{k}_{15}$ and $\bar{k}_{16}$ with k_{15} and k_{16}:

$$2\lambda\psi_{GN} = 2\lambda\psi_1 + (k_{15} - \bar{k}_{15})\kappa_{23}^2 + (k_{16} - \bar{k}_{16})\kappa_{13}^2. \tag{53}$$

In other words, this procedure results in the correct flexural stiffnesses, however, the torsional stiffnesses $\bar{k}_{12}$, $\bar{k}_{13}$, k_{14} predicted by this method are not in general agreement with k_{12}, k_{13} and k_{14}.

A difficulty associated with the Gibbs free energy procedure, is that the approximations (51) are not unique. Indeed, one could alternatively proceed with the approximations (46), however, the value of $\lambda\phi$ obtained using (46) will clearly not be the same as (52). As with the procedures discussed in Sections 4.2 and 4.3, recourse to three-dimensional solutions must also be made with this procedure. A final point of interest is that if one attempts to establish the expressions (50) corresponding to $\lambda\psi_{GN}$, then the results will be identical to (50) with $f = \mu/E$ and k_{12}, k_{13}, k_{14} replaced by $\bar{k}_{12}$, $\bar{k}_{13}$, $\bar{k}_{14}$. The resulting functions will not satisfy (47), in direct contrast to (51).

4.5. *The particular case of Bernoulli–Euler beam theory*

As noted by Naghdi and Rubin (1984), among others, the theory of the directed curve is sufficiently general that the Bernoulli–Euler beam theory may be developed as a constrained theory. For the Bernoulli–Euler beam theory, it is well known that there are several constraints on the functions $\mathbf{d}_\alpha$.† The linearization of these constraints for an initially straight rod implies that

$$\begin{aligned} &\gamma_{11} = 0, \quad \gamma_{12} = 0, \quad \gamma_{22} = 0, \\ &\kappa_{11} = 0, \quad \kappa_{22} = 0, \quad \kappa_{12} + \kappa_{21} = 0. \end{aligned} \tag{54}$$

The integrated expression for the strain-energy in this case may be obtained by substituting (54) into (42). Despite the resulting simplifications, the flexural rigidities are again given by $\bar{k}_{15}$ and $\bar{k}_{16}$ as opposed to the accepted values EI_α [cf. Love (1944, Ch. XVIII)]. Furthermore, the extensional stiffness is not the accepted value of EA, rather it is $\bar{k}_3$. A similar difficulty arises in connection with the torsional stiffness. Clearly, by directly integrating the three-dimensional strain-energy and imposing the constraints (54), the incorrect constitutive relations for the Bernoulli–Euler beam theory are obtained. If the method proposed in Section 4.2 were used, then $\lambda\psi_2$ would be specified as

$$2\lambda\psi_2 = E\left(1 - \frac{\mu(1-\nu)}{2(1-2\nu)}\right)\left(I_2\kappa_{13}^2 + I_1\kappa_{23}^2 + \frac{A}{4}\gamma_{33}^2\right) + (\mathscr{D} - \mu(I_1 + I_2))\kappa_{12}^2. \tag{55}$$

To motivate (55), we now proceed to discuss several other works on this theory.

First, Love's results, which are based on the earlier work of Kirchhoff and Clebsch, use the exact Saint-Venant solutions for flexure and torsion of a three-dimensional body and an approximation procedure for the state of strain. Recently, Dill (1992) has reexamined these works, and presents a clear derivation of the constitutive coefficients for this theory: $k_3 = EA$, $k_{12} + k_{13} = \mathscr{D}$, $k_{15} = EI_1$ and $k_{16} = EI_2$. This derivation does not employ an integration of the strain-energy.

The discrepancies associated with the direct integration procedure are also apparent from equations (3.7) and (3.26) of Daví (1992), who considers the rod as a model for a three-dimensional constrained elastic body. In Daví's paper, an integration procedure is used, which is similar to that recorded in the appendix, to establish the balance laws and

† Specifically, there are five constraints: $\mathbf{d}_\alpha = \mathbf{P}\mathbf{D}_\alpha$, $\mathbf{d} \cdot \mathbf{d}_3 = \mathbf{D}_\alpha \cdot \mathbf{D}_3$, where $\mathbf{P}$ is a proper orthogonal tensor. We do not include a detailed discussion of the indeterminate (or constraint) response functions here as they are readily available in the literature, see, e.g., Green and Laws (1973) or O'Reilly and Turcotte (1996).

constitutive relations for the rod. The discrepancy noted above, is removed by reinterpreting the coefficients in (42) and (43); a methodology proposed by Podio-Guidugli (1989).†

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† Daví's work is partially based on the methods of Podio-Guidugli (1989) and it is clear from his comments following eqn (3.26) that such a reinterpretation was used in Daví (1992).

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APPENDIX: THREE-DIMENSIONAL CONSIDERATIONS

We record here, from Green *et al.* (1974a, 1993), the specifications for the quantities associated with the directed curve in terms of the corresponding quantities in three-dimensional continuum mechanics.

We consider a three-dimensional body $\mathscr{B}$ and denote its fixed reference configuration and present configuration by $\boldsymbol{\kappa}_0$ and $\boldsymbol{\kappa}$, respectively. It is necessary to define a set of convected coordinates θ^i. The position vectors of a material point of $\mathscr{B}$ in these configurations are denoted by the respective vectors $\mathbf{R}^* = \mathbf{R}^*(\theta^i)$ and $\mathbf{r}^* = \mathbf{r}^*(\theta^i, t)$. Four sets of basis vectors for Euclidean three-space $\{\mathbf{g}_k^*\}$, $\{\mathbf{g}^{*k}\}$, $\{\mathbf{G}_k^*\}$ and $\{\mathbf{G}^{*k}\}$ are defined in the usual manner:

$$\mathbf{g}_k^* = \mathbf{g}_k^*(\theta^i, t) = \frac{\partial \mathbf{r}^*}{\partial \theta^k}, \quad \mathbf{g}^{*k}\cdot\mathbf{g}_i^* = \delta_i^k,$$

$$\mathbf{G}_k^* = \mathbf{G}_k^*(\theta^i) = \frac{\partial \mathbf{R}^*}{\partial \theta^k}, \quad \mathbf{G}^{*k}\cdot\mathbf{G}_i^* = \delta_i^k. \tag{A1}$$

The determinant of the metric tensors associated with two of these bases are denoted by $\sqrt{g^*} = \det(g_{ik}^*)$ and $\sqrt{G^*} = \det(G_{ik}^*)$, where $g_{ik}^* = \mathbf{g}_i^*\cdot\mathbf{g}_k^*$ and $G_{ik}^* = \mathbf{G}_i^*\cdot\mathbf{G}_k^*$. We also recall that the deformation gradient $\mathbf{F}^*$ of $\mathscr{B}$ has the representation $\mathbf{F}^* = \mathbf{g}_k^*\otimes\mathbf{G}^{*k}$ and that $J^* = \det(\mathbf{F}^*) = \sqrt{g^*}/\sqrt{G^*}$.

It is assumed that the body $\mathscr{B}$ is bounded by three material surfaces: $\mathrm{F} = \mathrm{F}(\theta^1, \theta^2, \xi = \theta^3) = 0$ and the two material surfaces $\xi = \theta^3 = \xi_\alpha$. In the integrals which follow, $\mathscr{A} = \mathscr{A}(\xi)$ is a material surface which corresponds to a coordinate surface $\xi =$ constant, and $\partial\mathscr{A} = \partial\mathscr{A}(\xi)$ is the material curve formed by the intersection of $\mathscr{A}$ with $F = 0$. It is assumed that the resulting curve is closed. To model the deformation of $\mathscr{B}$ using the directed curve, it is first necessary to choose the convected coordinates θ^i such that the following representation holds in the fixed reference configuration $\boldsymbol{\kappa}_0$ of $\mathscr{B}$:

$$\mathbf{R}^* = \mathbf{R}(\xi) + \theta^\alpha\mathbf{D}_\alpha(\xi), \tag{A2}$$

i.e. $\theta^3 = \xi$, $G_\alpha^* = \mathbf{D}_\alpha$ and $\mathbf{G}_3^* = \partial\mathbf{R}/\partial\xi + \theta^\alpha\partial\mathbf{D}_\alpha/\partial\xi$. We also note that

$$\mathbf{G}_k^*\otimes\mathbf{D}^k = \mathbf{I} + \theta^\alpha{}_0\mathbf{G}_\alpha, \quad \mathbf{G}^{*k}\otimes\mathbf{D}_k = (\mathbf{I} + \theta^\alpha{}_0\mathbf{G}_\alpha)^{-\mathrm{T}}. \tag{A3}$$

The tensor ${}_0\mathbf{G}_\alpha$ in (A3) is defined by $(4)_2$. Furthermore, $\mathbf{r}^*$ is approximated by (1).

To proceed, we recall the balance of linear and angular momenta and constitutive relations for an elastic body $\mathscr{B}$:

$$\left(\frac{\partial}{\partial\theta^i}(\mathbf{F}^*\mathbf{S}^*)\right)\mathbf{G}^{*i} + \rho_0^*\mathbf{f}^* = \rho_0^*\ddot{\mathbf{r}}^*, \quad \mathbf{S}^* = (\mathbf{S}^*)^{\mathrm{T}}, \quad \mathbf{S}^* = \rho_0^*\frac{\partial\psi^*}{\partial\mathbf{E}^*}. \tag{A4}$$

In (A4), $\mathbf{S}^*$ is the second (symmetric) Piola-Kirchhoff stress tensor, $\mathbf{f}^*$ is the body force per unit mass, ψ^* is the strain-energy function, and $\rho_0^* = \rho_0^*(\theta^i)$ is the mass density per unit volume. We consider solutions of (A4) of the form, [cf. (1), (7), (22) and (23)],

$$\mathbf{r}^* = \mathbf{r}(\xi, t) + \theta^\alpha\mathbf{d}_\alpha(\xi, t) \quad \mathbf{F}^* = \mathbf{g}_i^*\otimes\mathbf{G}^{i*} = \mathbf{F}(\mathbf{I} + \theta^\sigma\boldsymbol{\lambda}_\sigma)(\mathbf{I} + \theta^\alpha{}_0\mathbf{G}_\alpha)^{-1} \tag{A5}$$

where $\mathbf{F} = \mathbf{d}_i\otimes\mathbf{D}^i$. Substituting (A5) into (A4), it is easily seen that the following three equations may be obtained:

$$\left(\frac{\partial}{\partial \theta^i}(\mathbf{FAS^*})\right)\mathbf{G}^{*i}+\rho_0^*\mathbf{f}^* = \rho_0^*\ddot{\mathbf{r}}(\xi,t)+\theta^\alpha\rho_0^*\ddot{\mathbf{d}}_\alpha(\xi,t),$$

$$\theta^\beta\left(\frac{\partial}{\partial \theta^i}(\mathbf{FAS^*})\right)\mathbf{G}^{*i}+\rho_0^*\theta^\beta\mathbf{f}^* = \rho_0^*\theta^\beta\ddot{\mathbf{r}}(\xi,t)+\theta^\beta\theta^\alpha\rho_0^*\ddot{\mathbf{d}}_\alpha(\xi,t). \tag{A6}$$

These equations are used to establish partial differential equations for $\mathbf{r}$ and $\mathbf{d}_\alpha$.

By integrating (A6) over the region occupied by $\mathscr{B}$ in $\boldsymbol{\kappa}_0$, and then considering the partial derivative with respect to ξ, Green *et al.* (1970, 1974a, 1985, 1993) have shown that the resulting equations are respectively equivalent to the balance of linear momentum for the directed curve,

$$\frac{\partial \mathbf{n}}{\partial \xi}+\sqrt{D_{33}}\rho_0\mathbf{f} = \sqrt{D_{33}}\rho_0(\ddot{\mathbf{r}}+y^\alpha\ddot{\mathbf{d}}_\alpha), \tag{A7}$$

and the two balances of director momenta for the directed curve:

$$\frac{\partial \mathbf{m}^\alpha}{\partial \xi}+\sqrt{D_{33}}\rho_0\mathbf{l}^\alpha-\mathbf{k}^\alpha = \sqrt{D_{33}}\rho_0(y^\alpha\ddot{\mathbf{r}}+y^{\alpha\beta}\ddot{\mathbf{d}}_\beta). \tag{A8}$$

The balance of angular momentum for both theories is identically satisfied, if the constitutive relations are properly invariant under superposed rigid body motions. The constitutive relations discussed in this Appendix and Sections 2 and 3 are properly invariant and, consequently, the balance of angular momentum is not discussed.

The equivalence of (A7) to $(\text{A6})_1$ and (A8) to $(\text{A6})_2$ holds provided the fields in (A7) and (A8) are specified by, from the Appendix of Green and Naghdi (1993),

$$\lambda = \sqrt{D_{33}}\rho_0 = \int_{\mathscr{A}}\rho_0^*\sqrt{G^*}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2,\quad \lambda y^\alpha = \int_{\mathscr{A}}\rho_0^*\theta^\alpha\sqrt{G^*}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2,$$

$$\lambda y^{\alpha\beta} = \int_{\mathscr{A}}\rho_0^*\theta^\alpha\theta^\beta\sqrt{G^*}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2,$$

$$\mathbf{n} = \int_{\mathscr{A}}\mathbf{T}^3\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2,\quad \mathbf{m}^\alpha = \int_{\mathscr{A}}\mathbf{T}^3\theta^\alpha\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2,\quad \mathbf{k}^\alpha = \int_{\mathscr{A}}\mathbf{T}^\alpha\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2. \tag{A9}$$

i.e. from (8)–(10)

$$\sqrt{d_{33}}\mathbf{T} = \int_{\mathscr{A}}\mathbf{F^*S^*(F^*)}^{\mathrm{T}}\sqrt{G^*}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2,$$

$$\sqrt{d_{33}}\mathbf{M}^\alpha = \int_{\mathscr{A}}\mathbf{F^*S^*G}^{*3}\theta^\alpha\sqrt{G^*}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2\otimes\mathbf{d}_3. \tag{A10}$$

In writing (A9) and (A10), the following identities were used:

$$\mathbf{T}^{*i} = \sqrt{g^*}\mathbf{T}^*\mathbf{g}^{*i} = \sqrt{G^*}\mathbf{F^*S^*G}^{*i},$$

$$\mathbf{D}_3 = \frac{\partial \mathbf{R}}{\partial \xi},\quad D_{33} = \mathbf{D}_3\cdot\mathbf{D}_3,\quad \mathbf{d}_3 = \frac{\partial \mathbf{r}}{\partial \xi},\quad d_{33} = \mathbf{d}_3\cdot\mathbf{d}_3, \tag{A11}$$

where $\mathbf{T}^*$ is the Cauchy stress tensor. It remains to provide the expressions for the assigned force $\mathbf{f}$ and assigned director forces $\mathbf{l}^\alpha$ [from e.g. Green *et al.* (1974a)]:

$$\lambda\mathbf{f} = \int_{\mathscr{A}}\rho_0^*\mathbf{f}^*\sqrt{G^*}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2+\oint_{\partial\mathscr{A}}(\mathbf{T}^{*1}-\lambda'^1\mathbf{T}^{*3})\,\mathrm{d}\theta^2-(\mathbf{T}^{*2}-\lambda'^2\mathbf{T}^{*3})\,\mathrm{d}\theta^1,$$

$$\lambda\mathbf{l}^\alpha = \int_{\mathscr{A}}\rho_0^*\mathbf{f}^*\theta^\alpha\sqrt{G^*}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2+\oint_{\partial\mathscr{A}}\theta^\alpha(\mathbf{T}^{*1}-\lambda'^1\mathbf{T}^{*3})\,\mathrm{d}\theta^2-\theta^\alpha(\mathbf{T}^{*2}-\lambda'^2\mathbf{T}^{*3})\,\mathrm{d}\theta^1, \tag{A12}$$

where $\boldsymbol{\lambda}' = \lambda'^\alpha\mathbf{g}_\alpha^*+\mathbf{g}_3^*$ is a tangent vector to the surface $F = 0$. Clearly, the forces $\mathbf{f}$ and $\mathbf{l}^\alpha$ contain contributions from the three-dimensional body force $\mathbf{f}^*$ and the tractions on the lateral surface of the body.

In the discussion of Section 3.3, the following two quantities defined by Rubin (1996) are used:

$$\sqrt{D_{33}}A = \int_{\mathscr{A}}\sqrt{G^*}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2,\quad \sqrt{D_{33}}AA^\alpha = \sqrt{D}\int_{\mathscr{A}}\theta^\alpha\sqrt{G^*}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2, \tag{A13}$$

where $\sqrt{D} = [\mathbf{D}_1\mathbf{D}_2\mathbf{D}_3]$.